

Higher Order Oscillating Sequences, Affine Distal Flows on the d -Torus, and Sarnak's Conjecture ^{*†}

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Abstract

In this paper, we give two precise definitions of a higher order oscillating sequence and show the importance of this concept in the study of Sarnak's conjecture. We prove that any higher order oscillating sequence of order d is linearly disjoint from all affine distal flows on the d -torus for all $d \geq 2$. One consequence of this result is that any higher order oscillating sequence of order 2 is linearly disjoint from all affine flows on the 2-torus with zero topological entropy. In particular, this reconfirms Sarnak's conjecture for all affine flows on the 2-torus with zero topological entropy and for all affine distal flows on the d -torus for all $d \geq 2$.

1 Introduction

Suppose X is a compact metric space with metric $d(\cdot, \cdot)$. Let $T : X \rightarrow X$ be a continuous map. We call T a *flow* or a *dynamical system* because we will consider iterations $\{T^n\}_{n=0}^\infty$. Let \mathbb{C} denote the

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complex plane. Let \mathbb{R} , \mathbb{R}^2 , and \mathbb{R}^d denote the real line, the real plane, and the d -Euclidean space, respectively. We denote by \mathbb{N} the set of positive integers and by \mathbb{Z} the set of integers. Then \mathbb{Z}^2 and \mathbb{Z}^d are the integer lattices in \mathbb{R}^2 and \mathbb{R}^d . Let $C(X, \mathbb{C})$ be the space of all continuous functions $f : X \rightarrow \mathbb{C}$.

Suppose $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$ is a sequence of complex numbers. An important example is the Möbius sequence. Recall that the Möbius function $\mu(n)$ is, by definition, $\mu(n) = 1$ if $n = 1$; $\mu(n) = (-1)^r$ if $n = p_1 \cdots p_r$ for r distinct prime numbers $\{p_i\}_{i=1}^r$; $\mu(n) = 0$ if $p^2 | n$ for some prime number p . The Möbius sequence $\mathbf{u} = (\mu(n))_{n \in \mathbb{N}}$ is the one generated by the Möbius function. Following the idea of Sarnak [9, 10], we have the following definition.

Definition 1 (Disjointness). *We say the sequence $\mathbf{c} = (c_n)$ is linearly disjoint from the flow T if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N c_n f(T^n x) = 0 \quad (1)$$

for any $f \in C(X, \mathbb{C})$ and any $x \in X$.

Sarnak's conjecture (see [9, 10]) says that the Möbius sequence is linearly disjoint from all zero entropy flows. In [3], we introduce a new concept called an oscillating sequence for the purpose of the study of this conjecture.

Definition 2 (Oscillation). *The sequence $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$ is said to be an oscillating sequence if there is a constant $\lambda > 1$ such that*

$$K = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |c_n|^\lambda < \infty \quad (2)$$

and if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N c_n e^{2\pi i n t} = 0, \quad \forall 0 \leq t < 1. \quad (3)$$

We proved in [3] that any oscillating sequence is linearly disjoint from all minimally mean attractable (MMA) and minimally mean-L-stable (MMLS) flows. In the same paper, we further proved that flows defined by all p -adic polynomials of integral coefficients, all p -adic rational maps with good reduction, all automorphisms of the

2-torus with zero topological entropy, all diagonalizable affine maps of the 2-torus with zero topological entropy, all Feigenbaum maps, and all orientation-preserving circle homeomorphisms are MMA and MMLS. Due to Davenport's theorem [2], the Möbius sequence \mathbf{u} is an oscillating sequence. Therefore, we confirmed Sarnak's conjecture for these flows which form a large class of zero topological entropy flows. Recently, Huang, Wang, and Zhang [5] generalized a MLS flow to an ergodic flow with discrete spectrum for invariant measures and proved that Sarnak's conjecture holds for these flows.

However, consider

$$\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$$

the 2-torus. In [3], we also showed a counter-example as follows. Let

$$T_{A,\alpha} = A\mathbf{x}^t + \mathbf{a}^t$$

on the 2-torus \mathbb{T}^2 where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and $\mathbf{a} = (\alpha, 0) \in \mathbb{T}^2$ is a non-zero constant point and $\mathbf{x} = (x, y) \in \mathbb{T}^2$ is a variable. Here \mathbf{x}^t means the transpose of \mathbf{x} , that is,

$$\mathbf{x}^t = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then $T_{A,\alpha}$ is not MMLS on \mathbb{T}^d (it is MMA). But Liu and Sarnak in [6] and Wang in [11] showed that the Möbius sequence \mathbf{u} is linearly disjoint from this flows. Therefore, only the oscillation property is not enough for the purpose of the study of Sarnak's conjecture. We need the higher order oscillation property. There are two versions of a definition of the higher order oscillation (refer to [3, Remark 8]).

Definition 3 (Weaker Version of Higher Order Oscillation). *We call the sequence $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$ a higher order oscillating sequence of order $m \geq 2$ if it satisfies (2) and if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N c_n e^{2\pi i n^k t} = 0, \quad \forall 1 \leq k \leq m, \quad \forall 0 \leq t \leq 1. \quad (4)$$

Thanks to Hua's result [4], we knew that the Möbius sequence \mathbf{u} is a higher order oscillating sequence of order m for all $m \geq 2$ in this weaker version of the definition (Definition 3). Actually, according to [7], we have that for any $A > 0$,

$$\sum_{n=1}^N c_n e^{2\pi i n^k t} = O_A \left(N (\log N)^{-A} \right), \quad \forall 1 \leq k \leq m, \quad \forall 0 \leq t \leq 1. \quad (5)$$

Definition 4 (Stronger Version of Higher Order Oscillation). *We call the sequence $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$ a higher order oscillating sequence of order $m \geq 2$ if it satisfies (2) and if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N c_n e^{2\pi i P(n)} = 0 \quad (6)$$

for every real coefficient polynomial P of degree $\leq m$.

In their paper [6, Lemma 2.1], Liu and Sarnak showed that the Möbius sequence \mathbf{u} is also an higher order oscillating sequence of order m for all $m \geq 2$ in this stronger version of the definition (Definition 4). They actually showed an estimation like the one in (5).

In a recent work [1], we found another kind of higher order oscillating sequences of order m for any $m \geq 2$ in the stronger version of the definition (Definition 4), which is different from the one defined by an arithmetic function.

Theorem 1 (Uniform Distribution [1]). *Suppose g is a positive C^2 function on $(1, \infty)$ with non-negative first and second derivatives. For a fixed real number $\alpha \neq 0$ and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real number α), sequences*

$$\mathbf{c} = \left(e^{2\pi i \alpha \beta^n g(\beta)} \right)_{n \in \mathbb{N}}$$

are higher order oscillating sequence of order m for any $m \geq 2$ in the stronger version of the definition (Definition 4).

Remark 1. *In particular, when $g \equiv 1$, sequences in Theorem 1 are*

$$\mathbf{c} = \left(e^{2\pi i \alpha \beta^n} \right)_{n \in \mathbb{N}}.$$

Since the weaker version of the definition (Definition 3) is more nature for the oscillation property in ergodic theory, therefore, it becomes an interesting question that can the weaker version of the definition (Definition 3) implies the stronger version of the definition (Definition 4)? A proof of this will also provide another detailed proof of [6, Lemma 2.1].

In this paper, we first give a definition of a distal flow on the d -torus (Definition 5). And then prove our main result in this paper that any higher order oscillating sequence of order d is linearly disjoint from all affine distal flows on the d -torus. To present a clear idea, we first state and prove the main result for $d = 2$ (Theorem 2) and then prove a consequence that any higher order oscillating sequence of order 2 is linearly disjoint from all affine flows on the 2-torus with zero topological entropy (Corollary 1). In particular, this reconfirms Sarnak's conjecture for all affine flows on the 2-torus with zero topological entropy (Corollary 2). After that we state and prove the main result for $d > 2$. Thus it confirms Sarnak's conjecture for all affine distal flows on the d -torus (Corollary 3).

2 Statements of the Main Result

Suppose $d \geq 2$ is an integer. Let

$$\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$$

be the d -torus. Let $GL(d, \mathbb{Z})$ be the space of all $d \times d$ -matrices A of integer entries with determinants $\det(A) = \pm 1$. Let $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{T}^d$ be a variable and denote

$$\mathbf{x}^t = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$$

the transpose of \mathbf{x} . Then $A\mathbf{x}^t : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is an automorphism of \mathbb{T}^d . For $\mathbf{a} \in \mathbb{T}^d$, we have an affine map $A\mathbf{x}^t + \mathbf{a}^t : \mathbb{T}^d \rightarrow \mathbb{T}^d$. In order for this flow has entropy zero, the absolute values α of all eigenvalues of A must be all 1. If α is a complex number, then its complex conjugacy $\bar{\alpha}$ is also an eigenvalue. If $\alpha = e^{2\pi i\theta}$ for an irrational number θ , then the flow restricted on the union of the real part of the eigenspaces of α and $\bar{\alpha}$ is some kind rotation. Thus it is MMA and MMLS (see [3])

or the proof of Corollary 1 when $d = 2$). Most interesting dynamics of this map is on the eigenspaces of 1 and -1 . (If $\alpha = e^{2\pi i\theta}$ for a rational number $\theta = p/q$, then A^q has 1 or -1 as an eigenvalue.) In this case, except for the identity and the negative identity, by adding a non-zero shift $\mathbf{a} \in \mathbb{T}^d$, we get a distal flow, which has certain polynomial expansion but still keeps entropy zero. However, for the notational simplicity, we include the identity and the negative identity in the following definition.

Definition 5 (Affine Distal Flow). *We call an affine map*

$$T_{A,\mathbf{a}} = A\mathbf{x}^t + \mathbf{a}^t : \mathbb{T}^d \rightarrow \mathbb{T}^d$$

an affine distal flow if all eigenvalues of A are 1 (or -1) and $\mathbf{a} \neq \mathbf{0} \in \mathbb{T}^d$.

In the rest of the paper, we only use the stronger version of the definition (Definition 4). In order to present our idea more clearly, we divide our main result in two cases: $d = 2$ and $d > 2$. We state our main result in the case $d = 2$ first.

Theorem 2 (Main Theorem for $n = 2$). *Suppose $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$ is a higher order oscillating sequence of order 2. Then it is linearly disjoint from all affine distal flows $T_{A,\mathbf{a}}$ on the 2-torus \mathbb{T}^2 .*

One of the consequences of this main result is that

Corollary 1 (Zero Entropy for $n = 2$). *Suppose $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$ is a higher order oscillating sequence of order 2. Then it is linearly disjoint from all affine flows $T_{A,\mathbf{a}}$ on the 2-torus \mathbb{T}^2 with zero topological entropy.*

By combining Corollary 1 and [6, Lemma 2.1], this reconfirms Sarnak's conjecture for all affine flows with zero topological entropy.

Corollary 2 (Möbius Disjointness for $n = 2$). *The Möbius sequence $\mathbf{u} = (\mu(n))_{n \in \mathbb{N}}$ is linearly disjoint from all affine flows $T_{A,\mathbf{a}}$ on the 2-torus \mathbb{T}^2 with zero topological entropy.*

Now we state our main result in the case $d > 2$.

Theorem 3 (Main Theorem for Arbitrary $d > 2$). *For all $d > 2$, suppose $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$ is a higher order oscillating sequence of order d , then it is linearly disjoint from all affine distal flows $T_{A,\mathbf{a}}$ on the d -torus \mathbb{T}^d .*

By combining Theorem 3 and [6, Lemma 2.1], this confirms Sar-nak's conjecture for all affine distal flows on the d -torus \mathbb{T}^d .

Corollary 3 (Möbius Disjointness for $d > 2$). *For all $d > 2$, the Möbius sequence $\mathbf{u} = (\mu(n))_{n \in \mathbb{N}}$ is linearly disjoint from all affine distal flows on the d -torus \mathbb{T}^d .*

Our proof depends on the triangularization of an integral matrix. So we define a triangularizable affine distal flow. We say two affine distal flows $T_{A,\mathbf{a}}$ and $T_{B,\mathbf{b}}$ on the d -torus \mathbb{T}^d are topologically conjugate if there is a homeomorphism $h : \mathbb{T}^d \rightarrow \mathbb{T}^d$ such that

$$T_{A,\mathbf{a}} \circ h = h \circ T_{B,\mathbf{b}}.$$

Definition 6. *We say a distal flow $T_{A,\mathbf{a}}$ is triangularizable if it is topologically conjugate to a distal flow $T_{B,\mathbf{b}}$ such that $B = (b_{ij})_{d \times d}$ is a upper-triangle matrix, that is, $b_{ij} = 0$ for all $1 \leq j < i \leq d$, and $b_{ii} = 1$ for all $1 \leq i \leq d$ (or $b_{ii} = -1$ for all $1 \leq i \leq d$) and $b_{ij} \in \mathbb{Z}$ for all $1 \leq i < j \leq d$.*

More precisely, the $d \times d$ -matrix B in Definition 6 has the form

$$B = \pm \begin{pmatrix} 1 & b_{12} & b_{13} & \cdots & b_{1(d-1)} & b_{1d} \\ 0 & 1 & b_{21} & \cdots & b_{2(d-1)} & b_{2d} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & b_{(d-1)d} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad (7)$$

Before to prove Theorem 3, we will first prove the following lemma.

Lemma 1. *For any $d \geq 2$, any distal flow $T_{A,\mathbf{a}}$ on the d -torus \mathbb{T}^d is triangularizable.*

3 Proof of Theorem 2.

Consider the space $C(\mathbb{T}^2, \mathbb{C})$ of all complex continuous functions $f : \mathbb{T}^2 \rightarrow \mathbb{C}$ with the supreme norm,

$$\|f\| = \sup_{\mathbf{x} \in \mathbb{T}^2} |f(\mathbf{x})|.$$

Let $\mathbf{k} = (k, l) \in \mathbb{Z}^2$ and $\mathbf{x} = (x, y)$, define

$$e(\mathbf{k} \cdot \mathbf{x}) = e^{2\pi i(kx+ly)}, \quad \mathbf{x} \in \mathbb{T}^2.$$

Then $e(\mathbf{k} \cdot \mathbf{x}) \in C(\mathbb{T}^2, \mathbb{C})$. From the Stone-Weierstrass theorem (refer to [8]), we have that

Lemma 2. *The set*

$$S = \left\{ e(\mathbf{k} \cdot \mathbf{x}) \right\}_{\mathbf{k} \in \mathbb{Z}^2}$$

forms a dense subset in $C(\mathbb{T}^2, \mathbb{C})$.

A linear combination p of elements in S is called a trigonometric polynomial. We can write p as

$$p(\mathbf{x}) = \sum_{k_1 \leq k \leq k_2} \sum_{l_1 \leq l \leq l_2} a_{kl} e^{2\pi i(kx+ly)}.$$

Lemma 2 implies that for any $f \in C(\mathbb{T}^2, \mathbb{C})$, we have a sequence of trigonometric polynomials

$$p_q(\mathbf{x}) = \sum_{k_{1,q} \leq k \leq k_{2,q}} \sum_{l_{1,q} \leq l \leq l_{2,q}} a_{kl,q} e^{2\pi i(kx+ly)}. \quad (8)$$

such that $\|f - p_q\| \rightarrow 0$ as $q \rightarrow \infty$. The sequence $\{p_q\}_{q \in \mathbb{N}}$ is called the trigonometric approximation of f .

Given $f \in C(\mathbb{T}^2, \mathbb{C})$ and $\mathbf{x} \in \mathbb{T}^2$, let

$$S_N(\mathbf{x}) = \frac{1}{N} \sum_{n=1}^N c_n f(T_{A,\mathbf{a}}^n \mathbf{x}).$$

Let $\{p_q\}_{q \in \mathbb{N}}$ be a sequence of trigonometric polynomials approximating f in the supremum norm on $C(\mathbb{T}^2, \mathbb{C})$.

For any $\epsilon > 0$, we have an integer $r > 0$ such that

$$\|f - p_r\| < \frac{\epsilon}{2K^{\frac{1}{\lambda}}}.$$

Then

$$|S_N(\mathbf{x})| \leq \left| \frac{1}{N} \sum_{n=1}^N c_n |f(T_{A,\mathbf{a}}^n \mathbf{x}) - p_r(T_{A,\mathbf{a}}^n \mathbf{x})| \right| + \left| \frac{1}{N} \sum_{n=1}^N c_n p_r(T_{A,\mathbf{a}}^n \mathbf{x}) \right| = I + II.$$

For the estimation of I , we apply the Hölder inequality,

$$I \leq \left(\frac{1}{N} \sum_{l=1}^N |c_n|^\lambda \right)^{\frac{1}{\lambda}} \left(\frac{1}{N} \sum_{n=1}^N |f(T_{A,\mathbf{a}}^n \mathbf{x}) - p_r(T_{A,\mathbf{a}}^n \mathbf{x})|^{\lambda'} \right)^{\frac{1}{\lambda'}},$$

where $\lambda' > 1$ is the dual number of λ , that is, $1/\lambda + 1/\lambda' = 1$. Thus we have

$$I \leq K^{\frac{1}{\lambda}} \cdot \frac{\epsilon}{2K^{\frac{1}{\lambda}}} = \frac{\epsilon}{2}.$$

For the estimation of II , we first prove that $T_{A,\mathbf{a}}$ is triangularizable. Here we give a simple proof only working for $d = 2$ by using complex analysis. We give a proof for the general case in Lemma 1. However, the proof we will give below must be used interestingly in the last step in the proof of Lemma 1.

Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$$

and all eigenvalues of A are 1. (If all eigenvalues are -1 , then we consider $-A$.) It corresponds to the Möbius transformation

$$M(z) = \frac{az + b}{cz + d}, \quad z \in \mathbb{C}.$$

Since $\text{trace}(M) = 2$, it is a parabolic Möbius transformation. So it has only one fixed point

$$\frac{a-d}{2c} = \frac{p}{q}, \quad (p, q) = 1,$$

which is a rational point in \mathbb{C} . We have two integers r and s such that $pr - qs = 1$ because of the Bézout theorem.

Let

$$N(z) = \frac{pz + s}{qz + r}.$$

It corresponds to the invertible integral matrix

$$P = \begin{pmatrix} p & s \\ q & r \end{pmatrix} \in GL(2, \mathbb{Z})$$

with $\det(P) = 1$. The Möbius transformation N maps ∞ to p/q and has the inverse

$$N^{-1}(z) = \frac{rz - s}{-qz + p}$$

corresponding to the matrix

$$P^{-1} = \begin{pmatrix} r & -s \\ -q & p \end{pmatrix} \in GL(2, \mathbb{Z})$$

with $\det(P^{-1}) = 1$. Now consider the Möbius transformation $N^{-1} \circ M \circ N$. It is still a parabolic one and all coefficients are integers. Most important, it only fixes ∞ and has no other fixed point in \mathbb{C} . Therefore

$$N^{-1} \circ M \circ N(z) = z + t, \quad t \in \mathbb{Z},$$

which corresponds to the matrix

$$B = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{Z}. \quad (9)$$

Thus, we have that

$$P^{-1}AP = B.$$

The map

$$h(\mathbf{x}) = P\mathbf{x}^t : \mathbb{T}^2 \rightarrow \mathbb{T}^2$$

is a homeomorphism of the 2-torus \mathbb{T}^2 and conjugates $T_{A,\mathbf{a}}$ to $T_{B,\mathbf{b}}$ for $\mathbf{b} = P^{-1}\mathbf{a} \neq \mathbf{0} \in \mathbb{T}^2$, that is,

$$T_{A,\mathbf{a}} \circ h = h \circ T_{B,\mathbf{b}}.$$

Now let us continue to estimate II under the assumption that A is of the form (9). Suppose $\mathbf{a} = (a, b)$. For any $\mathbf{x} = (x, y) \in \mathbb{T}^2$, let

$$\mathbf{x}_n^t = T_{A,\mathbf{a}}^n \mathbf{x}^t = (x_n, y_n).$$

Due to the fact that A is of form (9), we have that

$$y_n = y + nb$$

and

$$x_n = x + (ty + a)n + tb\left(\sum_{j=1}^{n-1} j\right) = x + (ty + a)n + \frac{tb}{2}n(n-1).$$

Now using the formula (8) for p_r , we have that

$$p_r(T_{A,\mathbf{a}}^n \mathbf{x}) = \sum_{k_1 \leq k \leq k_2} \sum_{l_1 \leq l \leq l_2} a_{kl,r} e^{2\pi i P_{kl,r}(n)}$$

where

$$P_{kl,r}(n) = \frac{tbk}{2}n^2 + \left(k\left(ty + a - \frac{tb}{2}\right) + lb\right)n + (kx + ly)$$

is a real coefficient polynomial of degree 2 or 1 or 0.

Thus for the estimation of II , we have

$$\begin{aligned} II &= \left| \frac{1}{N} \sum_{n=1}^N c_n \sum_{k_1 \leq k \leq k_2} \sum_{l_1 \leq l \leq l_2} a_{kl,r} e^{2\pi i P_{kl,r}(n)} \right| \\ &= \left| \sum_{k_1 \leq k \leq k_2} \sum_{l_1 \leq l \leq l_2} a_{kl,r} \frac{1}{N} \sum_{n=1}^N c_n e^{2\pi i P_{kl,r}(n)} \right|. \end{aligned}$$

Let

$$L = \max\{|k_1|, |k_2|, |l_1|, |l_2|, |a_{kl,r}| \mid k_1 \leq k \leq k_2, l_1 \leq l \leq l_2\}.$$

Since $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$ is a higher order oscillating sequence of order 2, we can find an integer $M > r$ such that for $N > M$,

$$\left| \frac{1}{N} \sum_{n=1}^N c_n e^{2\pi i P_{kl,r}(n)} \right| < \frac{\epsilon}{2L^3}, \quad \forall k_1 \leq k \leq k_2, l_1 \leq l \leq l_2.$$

This implies that

$$II < \frac{\epsilon}{2}.$$

Therefore, we get that for all $N > M$,

$$|S_N(\mathbf{x})| < \epsilon.$$

This says that $\lim_{N \rightarrow \infty} S_N(\mathbf{x}) = 0$. We proved Theorem 2.

4 Proof of Corollary 1.

Let α and $\bar{\alpha}$ be two eigenvalues of A in the complex field \mathbb{C} and suppose $|\alpha| \geq 1$. The topological entropy of $T_{A,\mathbf{a}}$ is then $h(T_{A,\mathbf{a}}) = \log |\alpha|$. So $h(T_{A,\mathbf{a}}) = 0$ is equivalent to say that $|\alpha| = 1$. If $\lambda = e^{2\pi i\theta}$ for some $0 < \theta < 1$ but $\theta \neq 1/2$ (or when $\alpha = 1$ and the other eigenvalue is -1), then A is diagonalizable in the complex field \mathbb{C} . As we have proved [3, Proposition 8], $T_{A,\mathbf{a}}$ is an equicontinuous flow. Thus $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$ is linearly disjoint from $T_{A,\mathbf{a}}$ following our result in [3, Corollary 2] since a higher order oscillating sequence of order 2 is also an oscillating sequence. When all eigenvalues of A are 1 (or -1), then $T_{A,\mathbf{a}}$ is a distal flow. It is a consequence of Theorem 2. This completes the proof.

5 Proof of Lemma 1.

In the proof of Theorem 2, we already saw a complex analysis proof of that any distal flow on the 2-torus is triangularizable. But this proof only works for $d = 2$, although it is simple and neat. Here we give a proof for the general case. However, the argument we gave in the proof of Theorem 2 has to be used in the last step of this proof.

Suppose $d \geq 2$. Suppose all eigenvalues of A are 1. (If all eigenvalues are -1 , then we consider $-A$.) Since A is an integral matrix and 1 is its only eigenvalue, $A\mathbf{x}^t = \mathbf{x}^t$ has an integer solution. Suppose $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{Z}^d$ is a solution such that

$$\gcd(v_1, \dots, v_d) = 1.$$

Then, there are at least two of v_i , $1 \leq i \leq d$, which are relatively prime. Without loss of generality, we assume $(v_1, v_2) = 1$. From the Bézout theorem, there are two integers r and s such that $v_1 r - v_2 s = 1$. Consider the matrix

$$P_1 = \begin{pmatrix} v_1 & s & 0 & 0 & \cdots & 0 & 0 \\ v_2 & r & 0 & 0 & \cdots & 0 & 0 \\ v_3 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_d & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad (10)$$

Then $P_1 \in GL(d, \mathbb{Z})$ with $\det(P_1) = 1$.

Let $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$. We have that

$$A\mathbf{v}^t = \mathbf{v}^t \quad \text{and} \quad P_1\mathbf{e}_1^t = \mathbf{v}^t.$$

Thus,

$$P_1^{-1}\mathbf{v}^t = \mathbf{e}_1^t.$$

All these imply that

$$P_1^{-1}AP_1 = \begin{pmatrix} 1 & \mathbf{b}_{1(d-1)} \\ \mathbf{0}_{d-1}^t & A_1 \end{pmatrix} \quad (11)$$

where $\mathbf{0}_{d-1} = (0, \dots, 0) \in \mathbb{Z}^{d-1}$ and $\mathbf{b}_{1(d-1)} = (b_{12}, \dots, b_{1d}) \in \mathbb{Z}^{d-1}$ and $A_1 \in GL(d-1, \mathbb{Z})$ with $\det(A_1) = 1$. All eigenvalues of A_1 are 1.

Repeat the above argument for A_1 , we have a $\tilde{P}_1 \in GL(d-1, \mathbb{Z})$ with $\det(\tilde{P}_1) = 1$ such that

$$\tilde{P}_1^{-1}A_1\tilde{P}_1 = \begin{pmatrix} 1 & \mathbf{b}_{2(d-2)} \\ \mathbf{0}_{d-2}^t & A_2 \end{pmatrix}$$

where $\mathbf{0}_{d-2} = (0, \dots, 0) \in \mathbb{Z}^{d-2}$ and $\mathbf{b}_{2(d-2)} = (b_{23}, \dots, b_{3d}) \in \mathbb{Z}^{d-2}$ and $A_2 \in GL(d-2, \mathbb{Z})$ with $\det(A_2) = 1$. Let

$$P_2 = \begin{pmatrix} 1 & \mathbf{0}_{d-1} \\ \mathbf{0}_{d-1}^t & \tilde{P}_1 \end{pmatrix}.$$

Then we have

$$(P_1P_2)^{-1}AP_1P_2 = \begin{pmatrix} 1 & \mathbf{b}_{1(d-1)} \\ \mathbf{0}_{d-1}^t & \tilde{P}_1^{-1}A_1\tilde{P}_1 \end{pmatrix}.$$

Inductively, we obtain P_1, P_2, \dots, P_{d-2} such that

$$\hat{P} = P_1P_2 \cdots P_{d-2} \in GL(d, \mathbb{Z})$$

with $\det(\hat{P}) = 1$ and such that

$$\hat{P}^{-1}A\hat{P} = \begin{pmatrix} 1 & b_{12} & \cdots & b_{1(d-2)} & b_{1(d-1)} & b_{1d} \\ 0 & 1 & \cdots & b_{2(d-2)} & b_{2(d-1)} & b_{2d} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{(d-2)(d-1)} & b_{(d-2)d} \\ 0 & 0 & \cdots & 0 & a & b \\ 0 & 0 & \cdots & 0 & c & d \end{pmatrix}. \quad (12)$$

Let

$$A_{d-2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is a 2×2 -matrix in $GL(2, \mathbb{Z})$ with $\det(A) = 1$. All eigenvalues of A_{d-2} are 1. Now we apply the argument in the proof of Theorem 2 to get $\tilde{P}_{d-1} \in GL(2, \mathbb{Z})$ with $\det(\tilde{P}_{d-1}) = 1$ such that

$$\tilde{P}_{d-1}^{-1} A \tilde{P}_{d-1} = \begin{pmatrix} 1 & b_{(d-1)d} \\ 0 & 1 \end{pmatrix}, \quad b_{(d-1)d} \in \mathbb{Z}.$$

Define

$$P_{d-1} = \begin{pmatrix} I_{d-2} & 0 \\ 0 & \tilde{P}_{d-1} \end{pmatrix},$$

where I_{d-2} is the $(d-2) \times (d-2)$ identity matrix. And define

$$P = \hat{P} P_{d-1} = P_1 P_2 \cdots P_{d-2} P_{d-1}.$$

We finally get that $B = P^{-1} A P$ is of the form (7) with $+$. Let $\mathbf{b} = P^{-1} \mathbf{a} \in \mathbb{T}^d$. Define $h(\mathbf{x}) = P \mathbf{x}^t : \mathbb{T}^d \rightarrow \mathbb{T}^d$. It is a homeomorphism of the d -torus \mathbb{T}^d and we have that

$$T_{A, \mathbf{a}} \circ h = h \circ T_{B, \mathbf{b}}.$$

We completed the proof.

6 Proof of Theorem 3.

After the proof of Lemma 1, most of the proof of Theorem 3 is similar to that of Theorem 2 except for the notation and the estimation of II . However for the independence of two theorems, we run a full proof again.

Consider the space $C(\mathbb{T}^d, \mathbb{C})$ of all complex continuous functions $f : \mathbb{T}^d \rightarrow \mathbb{C}$ with the supreme norm,

$$\|f\| = \sup_{\mathbf{x} \in \mathbb{T}^d} |f(\mathbf{x})|.$$

Let $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$, define

$$e(\mathbf{k} \cdot \mathbf{x}) = e^{2\pi i(k_1 x_1 + \cdots + k_d x_d)}, \quad \mathbf{x} \in \mathbb{T}^d.$$

Then $e(\mathbf{k} \cdot \mathbf{x}) \in C(\mathbb{T}^d, \mathbb{C})$. From the Stone-Weierstrass theorem (refer to [8]), we have that

Lemma 3. *The set*

$$S = \left\{ e(\mathbf{k} \cdot \mathbf{x}) \right\}_{\mathbf{k} \in \mathbb{Z}^d}$$

forms a dense subset in $C(\mathbb{T}^d, \mathbb{C})$.

A linear combination p of elements in S is called a trigonometric polynomial. We can write p as

$$p(\mathbf{x}) = \sum_{m_1 \leq k_1 \leq s_1} \cdots \sum_{m_d \leq k_d \leq s_d} a_{\mathbf{k}} e^{2\pi i(k_1 x_1 + \cdots + k_d x_d)}.$$

Lemma 3 implies that for any $f \in C(\mathbb{T}^d, \mathbb{C})$, we have a sequence of trigonometric polynomials

$$p_q(\mathbf{x}) = \sum_{m_{1q} \leq k_1 \leq s_{1q}} \cdots \sum_{m_{dq} \leq k_d \leq s_{dq}} a_{\mathbf{k}, q} e^{2\pi i(k_1 x_1 + \cdots + k_d x_d)}. \quad (13)$$

such that $\|f - p_q\| \rightarrow 0$ as $q \rightarrow \infty$. The sequence $\{p_q\}_{q \in \mathbb{N}}$ is called the trigonometric approximation of f .

Given $f \in C(\mathbb{T}^d, \mathbb{C})$ and $\mathbf{x} \in \mathbb{T}^d$, let

$$S_N(\mathbf{x}) = \frac{1}{N} \sum_{n=1}^N c_n f(T_{A, \mathbf{a}}^n \mathbf{x}).$$

Let $\{p_q\}_{q \in \mathbb{N}}$ be a sequence trigonometric polynomials approximating f in the supremum norm on $C(\mathbb{T}^d, \mathbb{C})$.

For any $\epsilon > 0$, we have an integer $r > 0$ such that

$$\|f - p_r\| < \frac{\epsilon}{2K^{\frac{1}{\lambda}}}.$$

Then

$$|S_N(\mathbf{x})| \leq \left| \frac{1}{N} \sum_{n=1}^N c_n |f(T_{A, \mathbf{a}}^n \mathbf{x}) - p_r(T_{A, \mathbf{a}}^n \mathbf{x})| \right| + \left| \frac{1}{N} \sum_{n=1}^N c_n p_r(T_{A, \mathbf{a}}^n \mathbf{x}) \right| = I + II.$$

For the estimation of I , we apply the Hölder inequality,

$$I \leq \left(\frac{1}{N} \sum_{l=1}^N |c_l|^\lambda \right)^{\frac{1}{\lambda}} \left(\frac{1}{N} \sum_{n=1}^N |f(T_{A, \mathbf{a}}^n \mathbf{x}) - p_r(T_{A, \mathbf{a}}^n \mathbf{x})|^{\lambda'} \right)^{\frac{1}{\lambda'}},$$

where $\lambda' > 1$ is the dual number of λ , that is, $1/\lambda + 1/\lambda' = 1$. Thus we have

$$I \leq K^{\frac{1}{\lambda}} \cdot \frac{\epsilon}{2K^{\frac{1}{\lambda}}} = \frac{\epsilon}{2}.$$

For the estimation of II , we can assume A is of the form in (7) with $+$ due to Lemma 1.

Suppose $\mathbf{a} = (a_1, \dots, a_d)$. For any $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{T}^d$, let

$$\mathbf{x}_n^t = T_{A, \mathbf{a}}^n \mathbf{x}^t.$$

Denote $\mathbf{x}_n = (x_1^n, \dots, x_d^n) \in \mathbb{T}^d$.

To show a clear idea of our proof, we first assume $d = 3$. We have

$$x_3^n = x_3 + na_3,$$

$$x_2^n = x_2 + (b_{23}x_3 + a_2)n + b_{23}a_3 \sum_{j=1}^{n-1} j,$$

and

$$x_1^n = x_1 + (b_{12}x_2 + a_1)n + b_{12}(b_{23}x_3 + a_2) \sum_{j=1}^{n-1} j + b_{13}b_{23}a_3 \sum_{k=1}^{n-1} \sum_{j=1}^k j.$$

So we see that x_3^n is a polynomial of n of degree at most 1, x_2^n is a polynomial of n of degree at most 2, and x_1^n is a polynomial of n of degree at most 3.

In general, for $d > 2$, due to the fact that A is of form (7) with $+$, one can see that

$$x_d^n = x_d + na_d,$$

$$x_{d-1}^n = x_{d-1} + (b_{(n-1)n}x_n + a_{n-1})n + b_{(n-1)n}a_n \sum_{j_1=1}^{n-1} j_1.$$

In general

$$x_i^n = x_i^{n-1} + b_{i(i+1)}x_{i+1}^n + \dots + b_{id}x_d^{n-1}.$$

So for x_{d-2}^n , as in the case $d = 3$, it contains some single sums and a double sum $\sum_{j_2=1}^{n-1} \sum_{j_1=1}^{j_2} j_1$, which is a polynomial of degree 3. More general, suppose, in x_{d-j-1}^n , there is a term containing a degree j polynomial $p_j(n)$, then in x_{d-j}^n , there is a term containing

$$p_{j+1}(n) = \sum_{k=1}^{n-1} p_j(k).$$

which is a degree $j + 1$ polynomial. Thus we have that

$$\mathbf{x}_n = (P_{1d}(n), \dots, P_{d1}(n)),$$

where $P_{(d-j+1)j}(n)$ is a polynomial of degree at most j for each $1 \leq j \leq d$. Coefficients are all real and unchanged when $n > d$.

Now using the formula (13), we have that

$$p_r(T_{A,\mathbf{a}}^n \mathbf{x}) = \sum_{m_{1r} \leq k_1 \leq s_{1r}} \cdots \sum_{m_{dr} \leq k_d \leq s_{dr}} a_{\mathbf{k},r} e^{2\pi i P_{\mathbf{k},r}(n)},$$

where

$$P_{\mathbf{k},r}(n) = c_{\mathbf{k},r,d} n^d + \cdots + c_{\mathbf{k},r,1} n + c_{\mathbf{k},r,0}$$

is a real coefficient polynomial of degree at most d .

For the estimation of II , we have

$$\begin{aligned} II &= \left| \frac{1}{N} \sum_{n=1}^N c_n \sum_{m_{1r} \leq k_1 \leq s_{1r}} \cdots \sum_{m_{dr} \leq k_d \leq s_{dr}} a_{\mathbf{k},r} e^{2\pi i P_{\mathbf{k},r}(n)} \right| \\ &= \left| \sum_{m_{1r} \leq k_1 \leq s_{1r}} \cdots \sum_{m_{dr} \leq k_d \leq s_{dr}} a_{\mathbf{k},r} \frac{1}{N} \sum_{n=1}^N c_n e^{2\pi i P_{\mathbf{k},r}(n)} \right|. \end{aligned}$$

Let

$$L = \max\{|m_{1r}|, \dots, |m_{dr}|, |s_{1r}|, \dots, |s_{dr}|, |a_{\mathbf{k},r}| \mid m_{jr} \leq k_j \leq s_{jr}, 1 \leq j \leq d\}.$$

Since $\mathbf{c} = (c_n)_{n \in \mathbb{N}}$ is a higher order oscillating sequence of order d , we can find an integer $M > r$ such that for $N > M$,

$$\left| \frac{1}{N} \sum_{n=1}^N c_n e^{2\pi i P_{\mathbf{k},r}(n)} \right| < \frac{\epsilon}{2L^3}, \quad \forall m_{1r} \leq k_1 \leq s_{1r}, \dots, m_{dr} \leq k_d \leq s_{dr}.$$

This implies that

$$II < \frac{\epsilon}{2}.$$

Therefore, we get that for all $N > M$,

$$|S_N(\mathbf{x})| < \epsilon.$$

This says that $\lim_{N \rightarrow \infty} S_N(\mathbf{x}) = 0$. We proved Theorem 3.

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